RANDOM ARCS ON THE CIRCLE

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1. Introduction, Summary, and Historical Notes

The problem of coverage of the circle by a fixed number of randomly placed equal arcs has been considered by many investigators. In this paper, we present exact expressions for the moments of coverage of all orders, the cumulative distribution of coverage, and we give the limiting distribution of coverage as the number of arcs becomes large.

A recursive integral equation that expresses the moment of vacancy of order $m+1$ in terms of moments of vacancy of lower orders is given in theorem 1. This is the basic result from which the others follow. This equation is solved in theorem 2, giving the moments of vacancy and of coverage of all orders. A complete characterization of the distribution of the vacancy is given in theorem 3, and the distribution of the coverage then follows as a corollary. Finally, the asymptotic coverage distribution for fixed a as n tends to infinity is explored in theorem 4.

W. L. Stevens [10] derived an expression for the probability of complete coverage of the circle. C. Domb [5] found the coverage probability, the moments of coverage, and the distribution of coverage for the related problem in which the number of arcs has a Poisson distribution. He showed that the corresponding quantities for the problem of a fixed number of arcs could be found, at least in principle, by a series expansion. However, due to computational difficulties, he was unable to produce these formulae. D. A. Darling [4] treated aspects of this problem using characteristic functions. L. Flatto and

A. G. Konheim [7] explored the asymptotic behavior of the number of arcs at which complete coverage first occurs, as the arc length tends to zero. L. A. Shepp [9] also studied some asymptotics of this problem, as did P. J. Cooke [3]. G. Ailam [1] has provided a general mathematical framework within which to consider coverage problems.

2. Definitions

Let n arcs, each of length a, be placed independently with centers uniformly distributed over the circumference of a circle of length 1. Denote these random arcs by $X_1, \ldots, X_n$, the circumference of the circle by K, and Lebesgue measure on K by μ.

We define the coverage to be

$$(2.1) \qquad C_{(n,a)} = \mu(\cup_{i=1}^{n} X_i)$$

so that $C_{(n,a)}$ is the random proportion of the circumference that is contained in some arc. We define the vacancy to be

$$(2.2) \qquad D_{(n,a)} = \mu(\cap_{i=1}^{n} X_i^c) = 1 - C_{(n,a)}$$

where X_i^c denotes the complement of X_i in K,
so that $D_{(n,a)}$ is the random proportion of the circumference that is not contained in any arc. Note that $C_{(n,a)}$ and $D_{(n,a)}$ are random variables taking values in $[0,1]$. The moments of $C_{(n,a)}$ about zero are called moments of coverage. Those of $D_{(n,a)}$ are called moments of vacancy.

3. Results

Theorem 1: The moments of vacancy for n random arcs of length a on a circle satisfy the recursive integral equation

$$E\, D_{(n,a)}^{m+1} = \binom{m+n}{n}^{-1}(1-a)^{m+n}$$

$$(3.1) \qquad + m \sum_{k=1}^{n} \binom{n}{k} \int_{a}^{1-a} x^{m+k-1}(1-x-a)^{n-k} E\, D_{(k,a/x)}^{m} dx$$

$$+ m \int_{1-a}^{1} x^{m+n-1} E\, D_{(n,a/x)}^{m} dx$$

when $a < 1/2$, and

$$(3.2) \qquad E\, D_{(n,a)}^{m+1} = \binom{m+n}{n}^{-1}(1-a)^{m+n} + m \int_{a}^{1} x^{m+n-1} E\, D_{(n,a/x)}^{m} dx$$

when $a \geq 1/2$.

Proof of theorem 1: From (2.2), $D_{(n,a)}$ is the coverage of the random set $\bigcap_{i=1}^{n} X_i^c$, and we may use Robbins' [8] formula for its moments:

$$(3.3) \qquad E\, D_{(n,a)}^{m+1} = \int_{K^{m+1}} P(u_1,\ldots,u_{m+1} \in \bigcap_{i=1}^{n} X_i^c)\, du_1 \cdots du_{m+1} .$$

Since $X_1,\ldots,X_n$ are independent and identically distributed sets,

$$(3.4) \qquad P(u_1,\ldots,u_{m+1} \in \bigcap_{i=1}^{n} X_i^c) = [P(u_1,\ldots,u_{m+1} \in X_1^c)]^n .$$

Using invariance of the integrand under permutations of $(u_1,\ldots,u_{m+1})$ with rotational symmetry, (3.3) and (3.4) may be written as

$$(3.5) \qquad E\, D^{m+1}_{(n,a)} = m! \int_{\{0 \le u_1 \le \cdots \le u_m \le 1\}} [P(u_1,\ldots,u_m,1 \epsilon X_1^c]^n du_1 \ldots du_m \quad .$$

If the random arc X_1 is to contain none of the ordered points $u_1,\ldots,u_m,1,$ then it must be between a pair of them. Thus

$$(3.6) \quad P(u_1,\ldots,u_m,1 \epsilon X_1^c) = (u_1-a)_+ + (u_2-u_1-a)_+ + \cdots + (u_m-u_{m-1}-a)_+ + (1-u_m-a)_+$$

where $(t)_+$ denotes the larger of t and zero. The crucial inductive step is to observe that from (3.6) it follows that

$$(3.7) \quad P(u_1,\ldots,u_m,1 \epsilon X_1^c) = u_m P(\frac{u_1}{u_m},\ldots,\frac{u_{m-1}}{u_m},1 \epsilon Y_1^c) + (1-u_m-a)_+$$

where Y_1 denotes a random arc of length a/u_m. Substituting (3.7) into (3.5) and changing variables to $v_i = u_i/u_m,$ we see that

$$(3.8) \quad E\, D^{m+1}_{(n,a)} = m! \int_0^1 u_m^{m-1} \int_{\{0 \le v_1 \le \cdots \le v_{m-1} \le 1\}} \{u_m P(v_1,\ldots,v_{m-1},1 \epsilon Y_1^c) + (1-u_m-a)_+\}^n dv_1 \ldots dv_{m-1} \cdot du_m$$

Considering the cases $u_m \le a,\ u_m \le 1-a,\ u_m > 1-a,$ and expanding the integrand, we find that for $a < 1/2$:

$$
(3.9)\quad
\begin{aligned}
E\, D^{m+1}_{(n,a)} &= m \int_0^a u_m^{m-1}(1-u_m-a)^n du_m \\
&+ m!\int_a^{1-a} u_m^{m-1} \int_{\{0 \le v_1 \le \cdots \le v_{m-1} \le 1\}} \{u_m P(v_1,\ldots,v_{m-1},1\epsilon Y_1^c)+(1-u_m-a)_+\}^n dv_1 \ldots dv_{m-1} du_m \\
&+ m!\int_a^1 u_m^{m+n-1} \int_{\{0 \le v_1 \le \cdots \le v_{m-1} \le 1\}} [P(v_1,\ldots,v_{m-1}\epsilon Y_1^c)]^n dv_1 \ldots dv_{m-1} du_m \quad .
\end{aligned}
$$

We recognize the final, inner integral to be $E\, D^m_{(n,a/u_m)}/(m-1)!$ Expanding the integrand of the second integral, we recognize terms of the form $E\, D^m_{(k,a/u_m)}$. If we also perform the first beta integral and substitute x for u_m, we have (3.1), completing the proof when $a \le 1/2$.

If $a > 1/2$, then instead of (3.9) we have

$$
(3.10)\quad
\begin{aligned}
E\, D^{m+1}_{(n,a)} &= m\int_0^a u_m^{m-1}(1-u_m-a)^n du_m \\
&+ m!\int_a^1 u_m^{m+n-1} \int_{\{0 \le v_1 \le \cdots \le v_{m-1} \le 1\}} [P(v_1,\ldots,v_{m-1},1\epsilon Y_1^c)]^n dv_1 \ldots dv_{m-1} du_m \quad .
\end{aligned}
$$

Changing variables to $x = u_m/(1-a)$ in the first integral to obtain a beta integral, evaluating this, observing that the inner second integral is $E\, D^m_{(n,a/u_m)}/(m-1)!$, and substituting x for u_m, we obtain (3.2) and the proof is complete. ||

<u>Theorem 2:</u> The moments of vacancy for n random arcs of length a on a circle are given by

$$(3.11) \qquad E\, D^m_{(n,a)} = \binom{m+n-1}{n}^{-1} \sum_{\ell=1}^{m} \binom{m}{\ell}\binom{n-1}{\ell-1}(1-\ell a)_+^{m+n-1}, \qquad m \geq 1$$

where $(t)_+$ denotes the larger of t and zero. Moments of coverage are therefore

$$(3.12) \qquad E\, C^m_{(n,a)} = 1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} E\, D^k_{(n,a)}$$

Proof of theorem 2: The proof is by induction on m, using the recursion formulae of theorem 1. Begin by observing that when $m = 1$, (3.11) yields

$$(3.13) \qquad E\, D_{(n,a)} = (1-a)^n$$

which may be verified directly using Robbins' theorem. It remains only to show that (3.11) satisfies the proper recursion formula, (3.1) or (3.2), depending on the value of a.

When $a < 1/2$, using the induction hypothesis and substituting $E\, D^m_{(k,a/x)}$ from (3.11) into (3.1) we have

$$E\, D^{m+1}_{(n,a)} = \binom{m+n}{n}^{-1}(1-a)^{m+n}$$

$$(3.14) \qquad + m \sum_{k=1}^{n} \binom{n}{k}\binom{m+k-1}{k}^{-1} \sum_{\ell=1}^{m} \binom{m}{\ell}\binom{k-1}{\ell-1} \int_a^{1-a} (1-x-a)^{n-k}(x-\ell a)_+^{m+k-1}dx$$

$$+ m\binom{m+n-1}{n}^{-1} \sum_{\ell=1}^{m} \binom{m}{\ell}\binom{n-1}{\ell-1} \int_{1-a}^{1} (x-\ell a)_+^{m+n-1}dx \quad .$$

The first integral may be done using a change of variables as follows:

$$\int_a^{1-a} (1-x-a)^{n-k}(x-\ell a)_+^{m+k-1}dx = [\int_{\ell a}^{1-a} (1-x-a)^{n-k}(x-\ell a)^{m+k-1}dx]I^{\{\ell a < 1-a\}}$$

$$(3.15) \qquad = [1-(\ell+1)a]_+^{m+n} \int_0^1 (1-x-a)^{n-k}x^{m+k-1}dx$$

$$= [1-(\ell+1)a]_+^{m+n}(m+k)^{-1}\binom{m+n}{m+k}^{-1}$$

where $I^{\{\ell a<1-a\}}=1$ if $\ell a<1-a$ and is 0 otherwise.

Substituting this into (3.14), interchanging the order of summation, simplifying, and using the fact that

$$(3.16) \qquad \sum_{k=\ell}^{n} \binom{k-1}{\ell-1} = \binom{n}{\ell}$$

we obtain

$$E\, D_{(n,a)}^{m+1} = \binom{m+n}{n}^{-1}\{(1-a)^{m+n}$$

$$(3.17) \qquad + \sum_{\ell=1}^{m} \binom{m}{\ell}\binom{n}{\ell}[1-(\ell+1)a]_+^{m+n}$$

$$+ \sum_{\ell=1}^{m} \binom{m}{\ell}\binom{n-1}{\ell-1}[(1-\ell a)_+^{m+n}-(1-(\ell+1)a)_+^{m+n}]\}.$$

Gathering coefficients of $(1-\ell a)_+^{m+n}$ and simplifying, we obtain

$$(3.18) \qquad E\, D_{(n,a)}^{m+1} = \binom{m+n}{n}^{-1} \sum_{\ell=1}^{m+1} \binom{m+1}{\ell}\binom{n-1}{\ell-1}(1-\ell a)_+^{m+n} ,$$

completing the proof for the case $a < 1/2$.

When $a \geq 1/2$, (3.11) reduces to

$$E\, D^m_{(n,a)} = m\binom{m+n-1}{n}^{-1}(1-a)^{m+n-1} \tag{3.19}$$

and it is straightforward to verify that this satisfies the recursion formula (3.2).

Moments of coverage (3.12) are easily found using the binominal expansion and (2.2). ||

We are now in a position to give a complete description of the distribution of the vacancy $D_{(n,a)}$ and of the coverage $C_{(n,a)}$.

Theorem 3: The vacancy $D_{(n,a)}$ of n random arcs of length a on a circle may be expressed as a mixture of a degenerate and a continuous random variable:

$$D_{(n,a)} = \begin{cases} A_{(n,a)} & \text{probability} \quad p_{(n,a)} \\ B_{(n,a)} & \qquad\qquad\quad 1-p_{(n,a)} \end{cases} \tag{3.20}$$

where $A_{(n,a)} \equiv (1-na)_+$ is degenerate and $B_{(n,a)}$ is continuous with density

$$f_{(n,a)}(t) = \frac{n}{1-p_{(n,a)}} \sum_{\ell=1}^{n} \sum_{k=1}^{n-1} (-1)^{k+\ell}\binom{n-1}{\ell-1}\binom{n-1}{k}\binom{\ell-1}{k-1} t^{k-1}(1-\ell a-t)_+^{n-k-1} \tag{3.21}$$

subject to the convention that

$$(3.22) \qquad (1-\ell a-t)_+^0 = \begin{cases} 1 & \text{if } 1-\ell a-t \geq 0 \\ 0 & \phantom{\text{if }} 1-\ell a-t < 0 \end{cases} .$$

The mixing probability is

$$(3.23) \qquad p_{(n,a)} = \begin{cases} \sum_{\ell=0}^{n} (-1)^{\ell}\binom{n}{\ell}(1-\ell a)_+^{n-1} & \text{if } na > 1 \\ (1-na)^{n-1} & \phantom{\text{if }} na \leq 1 \end{cases}$$

and the cumulative distribution function of $D_{(n,a)}$ is

$$(3.24) \quad F_{(n,a)}(t) = P(D_{(n,a)} \leq t) = 1 + \sum_{\ell=1}^{n} \sum_{k=0}^{n-1} (-1)^{k+\ell}\binom{n}{\ell}\binom{\ell-1}{k}\binom{n-1}{k} t^k (1-\ell a-t)_+^{n-k-1} .$$

Some cases of $f_{(n,a)}(t)$ and $F_{(n,a)}(t)$ are plotted in figures 3.1 through 3.6.

The proof of this theorem will follow from the following technical lemmas. The first lemma established the decomposition (3.20) with $A_{(n,a)}$ degenerate and $p_{(n,a)}$ given by (3.23).

<u>Lemma 1</u>: $D_{(n,a)}$ has mass at least $p_{(n,a)}$ (given by (3.23)) at $(1-na)_+$, and $p_{(n,a)} > 0$ unless $na = 1$.

<u>Proof</u>: We consider three cases. First, if $a > 1/n$, then $(1-na)_+ = 0$ and $p_{(n,a)} = \Sigma_{\ell=0}^{n} (-1)^{\ell}\binom{n}{\ell}(1-\ell a)_+^{n-1}$. In this case, $na > 1$ and the circle will be covered with positive probability. This probability was found by Stevens [10] to be $p_{(n,a)}$. Since $D_{(n,a)} = 0$ is the event that the circle is covered, the lemma holds in this case.

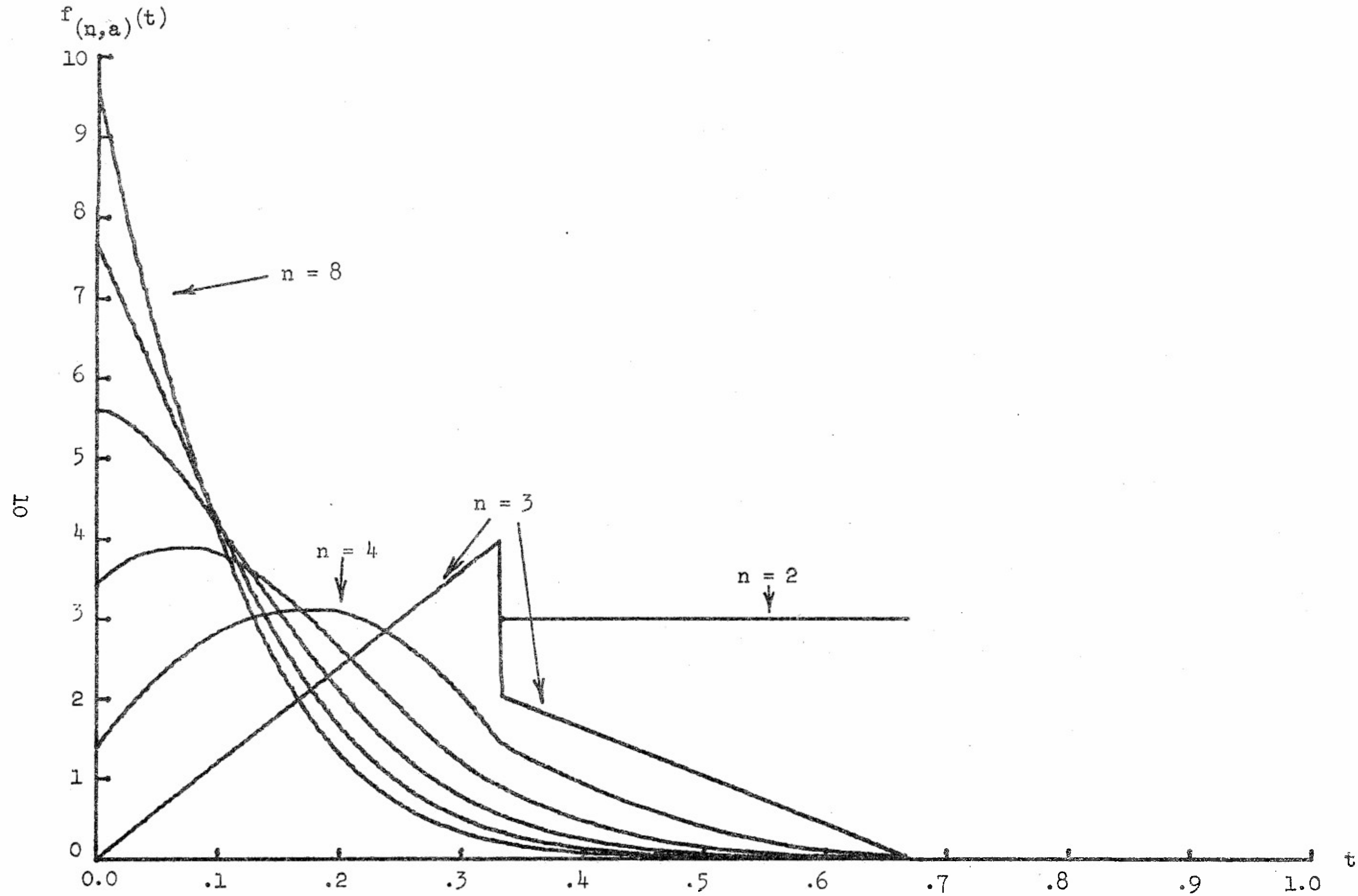

Figure 3.1: Density of $B_{(n,a)}$ in the case of n arcs of length $a = 1/4$, $a = 1/3$, for $n = 2$ through 8.

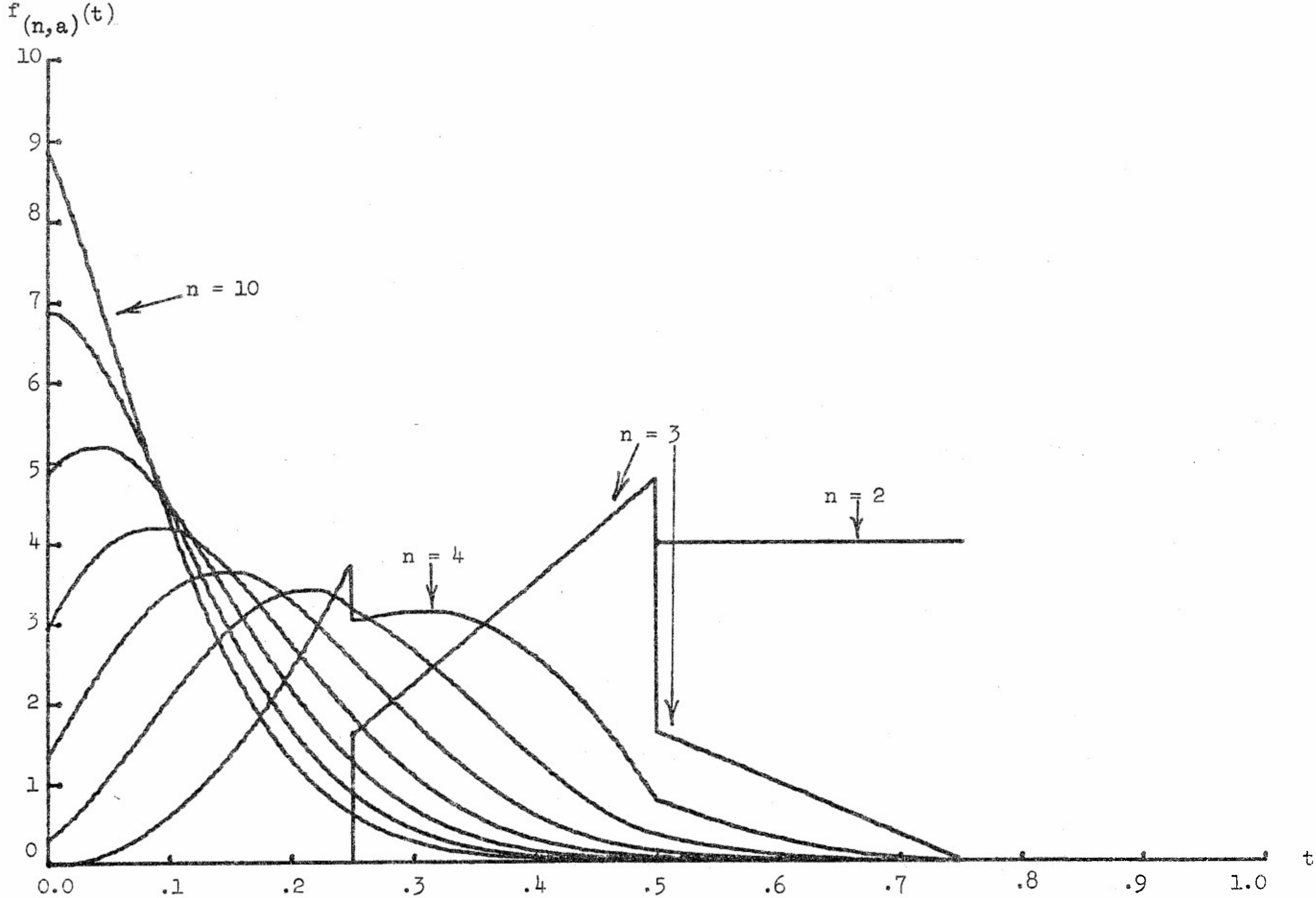

Figure 3.2: Density of $B_{(n,a)}$ in the case of n arcs of length $a = 1/4$, for $n = 2$ through 10.

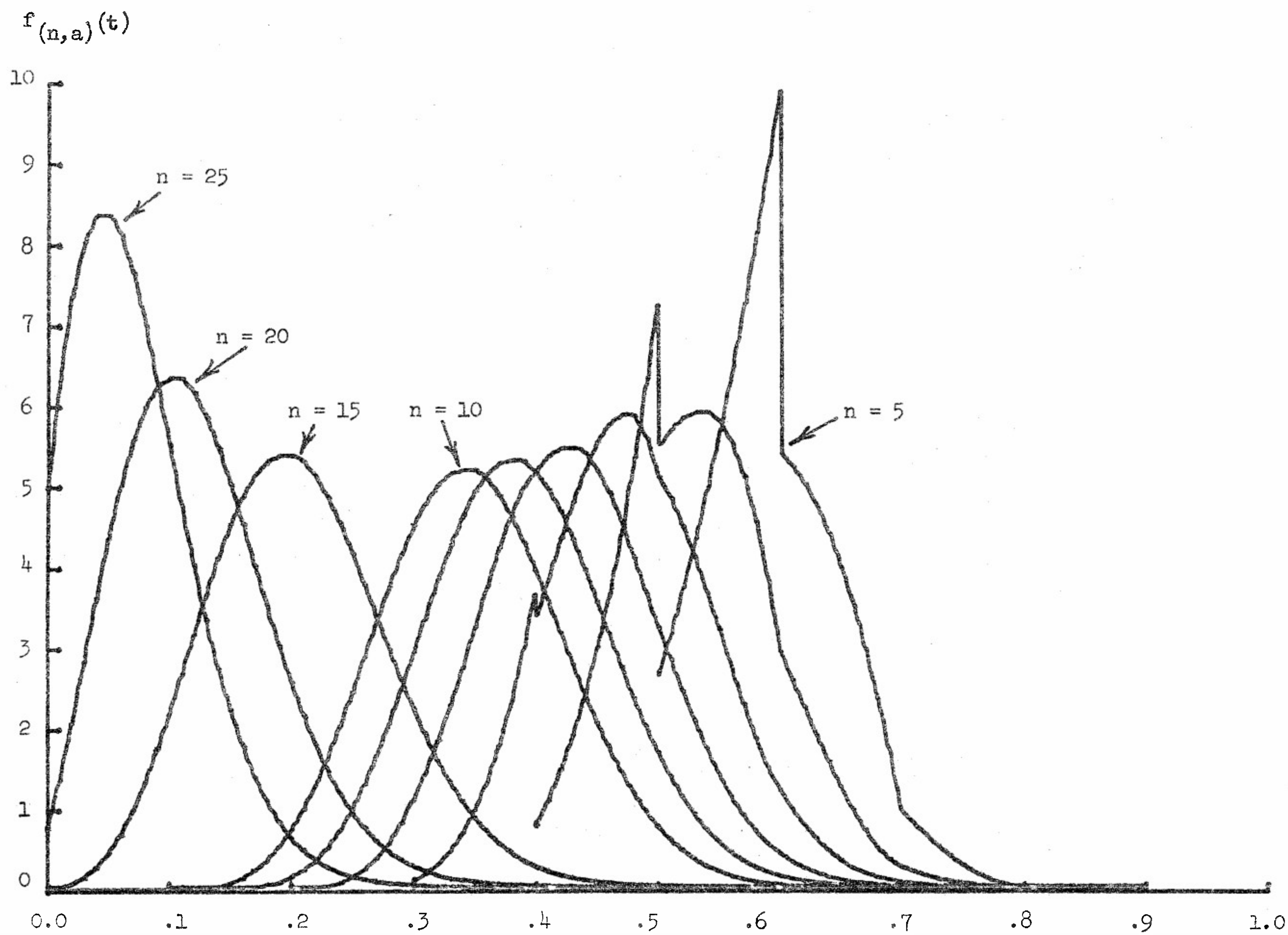

Figure 3.3: Density of $B_{(n,a)}$ in the case of n arcs of length $a = 1/10$, for $n = 5,6,7,8,9,10,15,20$, and 25.

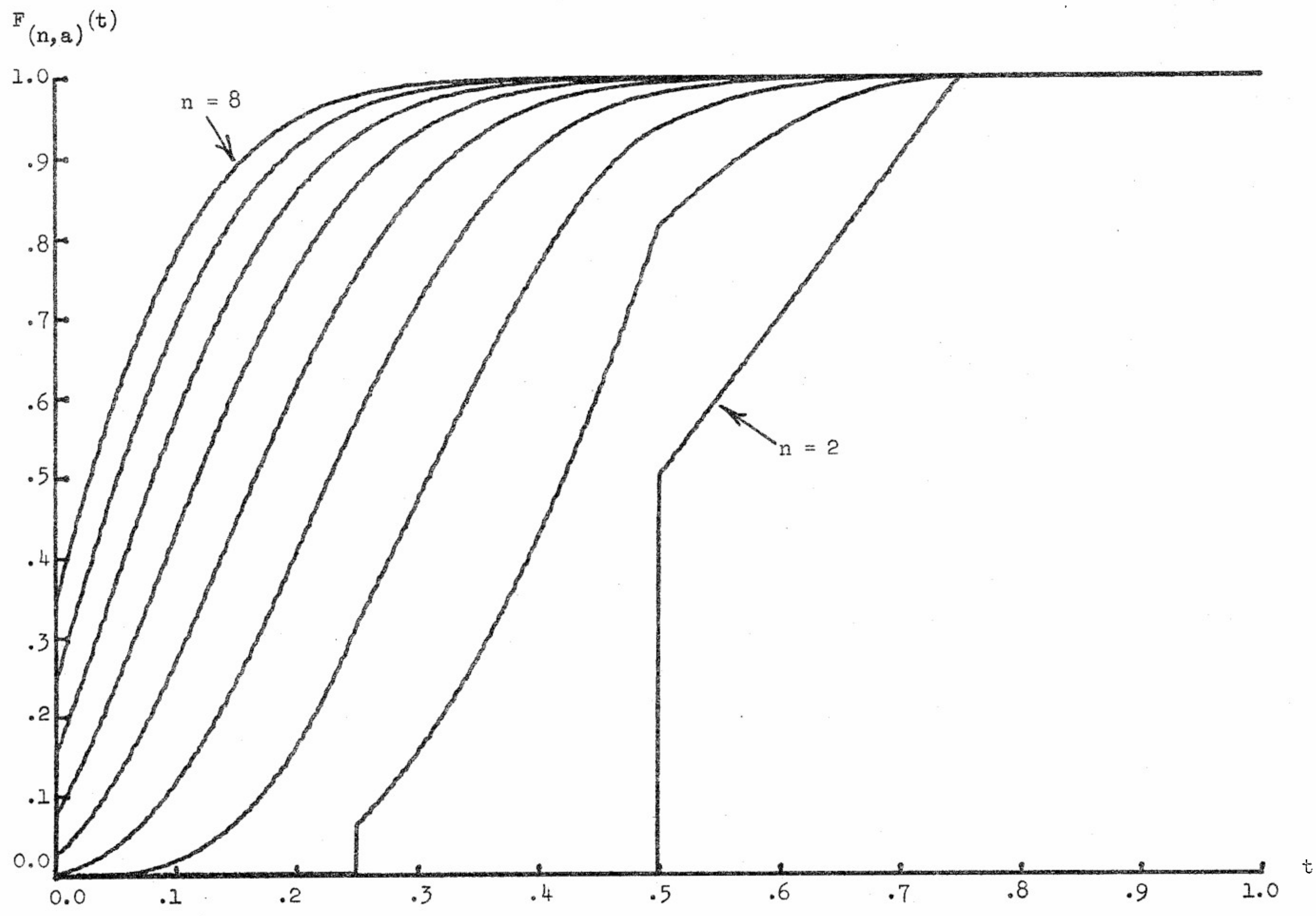

Figure 3.4: Cumulative distribution of $D_{(n,a)}$ in the case of n arcs of length $a = 1/3$, for $n = 2$ through 8.

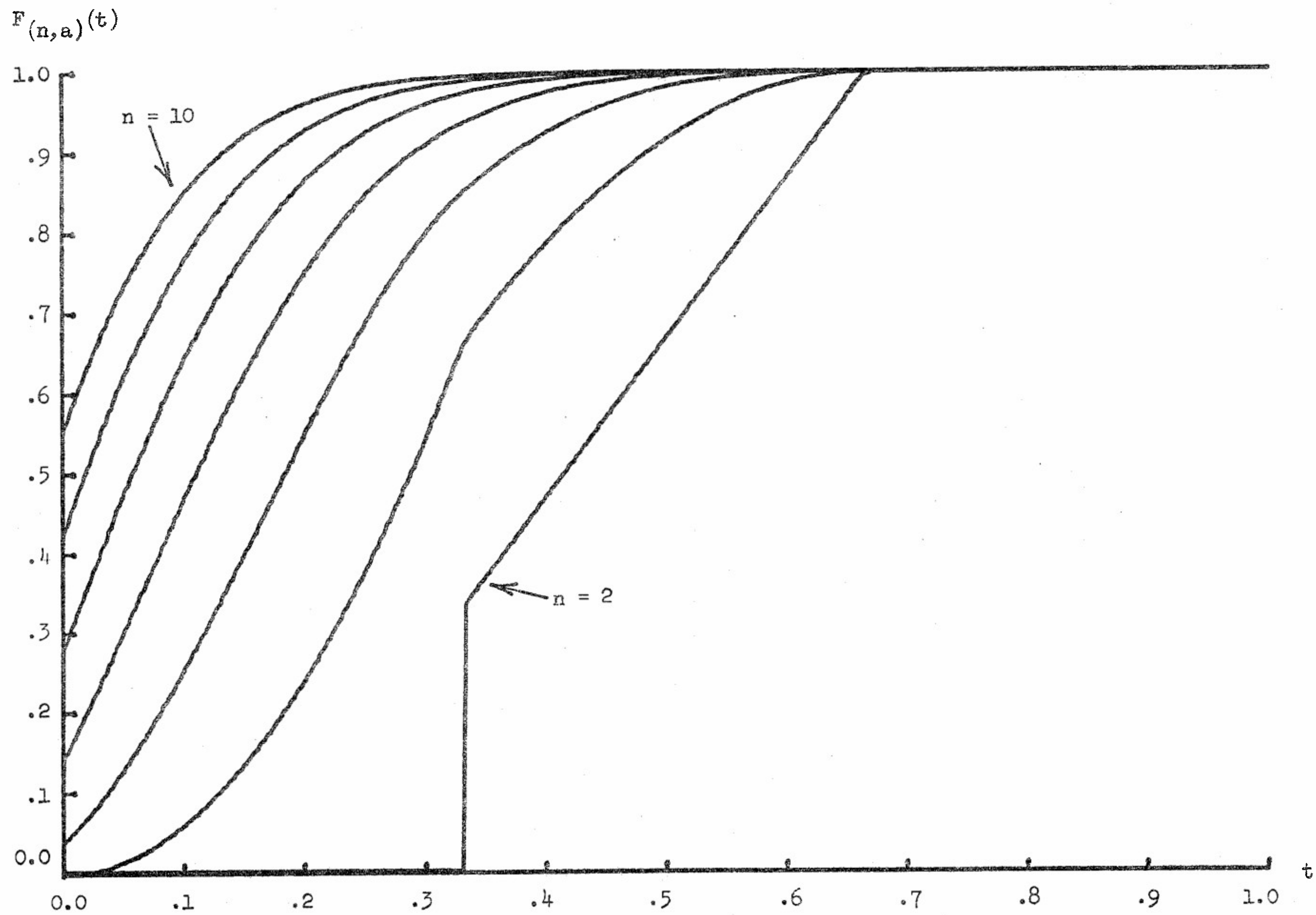

Figure 3.5: Cumulative distribution of $D_{(n,a)}$ in the case of n arcs of length $a = 1/4$, for $n = 2$ through 10.

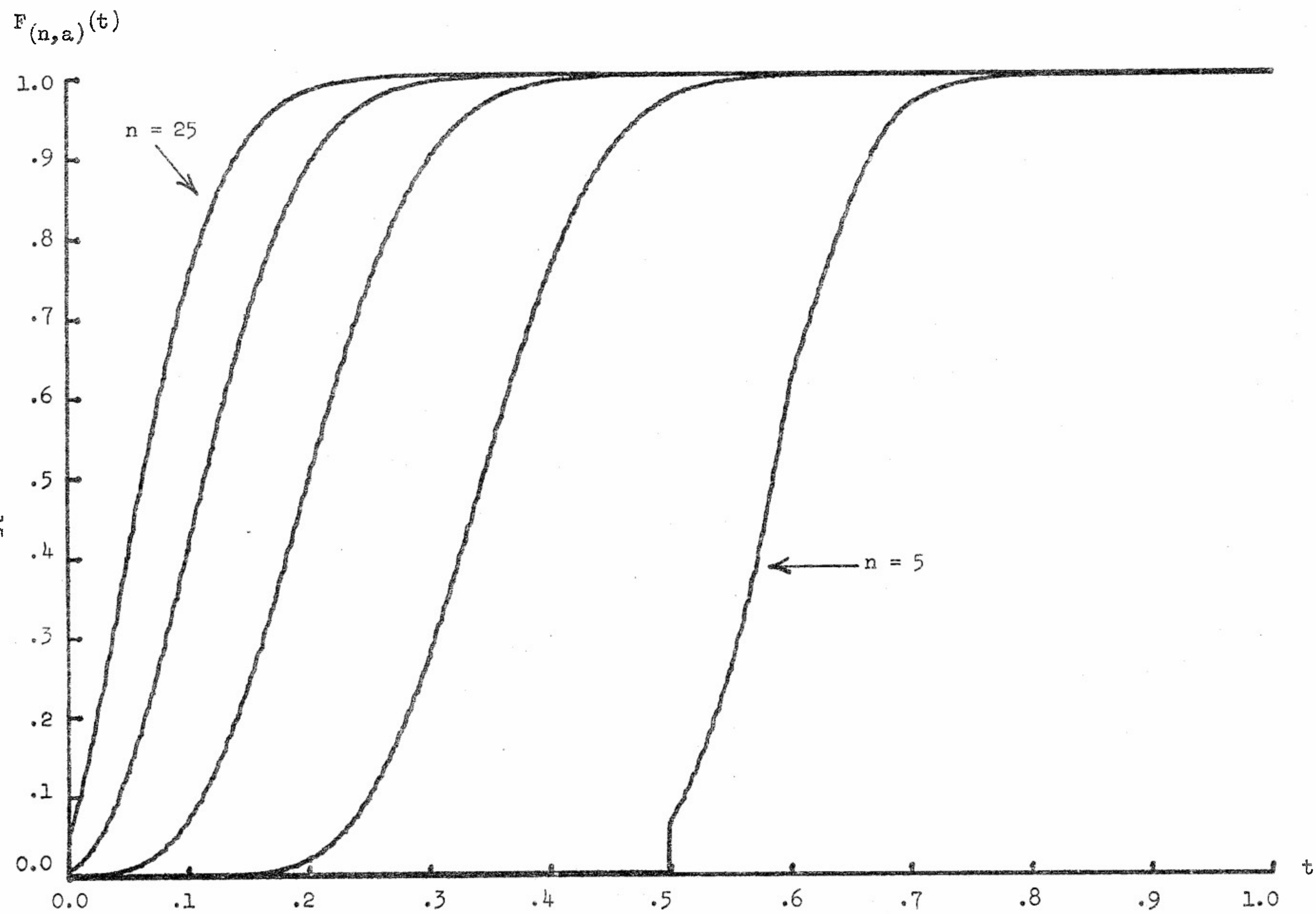

Figure 3.6: Cumulative distribution of $D_{(n,a)}$ in the case of n arcs of length $a = 1/10$, for $n = 5,10,15,20,$ and 25.

Next, consider the case where $a < 1/n$, so that $(1-na)_+ = 1-na$ and $p_{(n,a)} = (1-na)^{n-1}$. In this case, $na < 1$ and with positive probability none of the arcs will overlap, an event equivalent to $D_{(n,a)} = 1-na$. This probability is shown to be this value of $p_{(n,a)}$ in Feller volume II [6], problem 22 of chapter I. Thus the lemma is true in this case as well.

Finally, if $na = 1$, then $p_{(n,a)} = 0$ and the lemma is trivially true. ||

Lemma 2: The moments of $B_{(n,a)}$ are the same as the moments of $f_{(n,a)}$, as defined by (3.21).

Proof: First we calculate the moments of $B_{(n,a)}$. From the decomposition (3.20) we have

$$E\, D^m_{(n,a)} = p_{(n,a)}(1-na)^m_+ + (1-p_{(n,a)})E B^m_{(n,a)} \tag{3.25}$$

so that

$$E\, B^m_{(n,a)} = \frac{1}{1-p_{(n,a)}} E\, D^m_{(n,a)} - \frac{p_{(n,a)}}{1-p_{(n,a)}}(1-na)^m_+ \tag{3.26}$$

Substituting for $E\, D^m_{(n,a)}$ from (3.11) and for $p_{(n,a)}$ from (3.23) we have

$$E\, B^m_{(n,a)} = \frac{1}{1-p_{(n,a)}} \binom{m+n-1}{n}^{-1} \sum_{\ell=1}^{m} \binom{m}{\ell}\binom{n-1}{\ell-1}(1-\ell a)^{m+n-1}_+ - \frac{(1-na)^{m+n-1}_+}{1-p_{(n,a)}} . \tag{3.27}$$

Now we let

$$(3.28) \qquad \zeta_{(n,a,m)} = \int_0^1 t^m f_{(n,a)}(t)dt$$

denote the m^{th} moment of the function $f_{(n,a)}$. Substituting for $f_{(n,a)}(t)$ from (3.21) we have

$$(3.29) \quad \zeta_{(n,a,m)} = \frac{n}{1-p_{(n,a)}} \sum_{\ell=1}^{n} \sum_{k=1}^{n-1} (-1)^{k+\ell} \binom{n-1}{\ell-1}\binom{n-1}{k}\binom{\ell-1}{k-1} \int_0^1 t^{m+k-1}(1-\ell a-t)_+^{n-k-1}dt \ .$$

The integral may be evaluated by changing variables and performing a beta integral as follows:

$$(3.30) \quad \begin{aligned} \int_0^1 t^{m+k-1}(1-\ell a-t)_+^{n-k-1}dt &= \int_0^{(1-\ell a)_+} t^{m+k-1}(1-\ell a-t)^{n-k-1}dt \\ &= (1-\ell a)_+^{m+n-1} \int_0^1 t^{m+k-1}(1-t)^{n-k-1}dt = (1-\ell a)_+^{m+n-1} \frac{(m+k-1)!(n-k-1)!}{(m+n-1)!} \end{aligned}$$

Substituting this into (3.29) and simplifying, we have

$$(3.31) \quad \zeta_{(n,a,m)} = \frac{n}{1-p_{(n,a)}} \binom{m+n-1}{m}^{-1} \sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell} \binom{n-1}{\ell-1}(1-\ell a)_+^{m+n-1} \sum_{k=1}^{n-1} (-1)^k \binom{\ell}{k}\binom{m+k-1}{k-1} \ .$$

It is not difficult to show that

$$(3.32)\quad \sum_{k=1}^{n-1}(-1)^k\binom{\ell}{k}\binom{m+k-1}{k-1} = \begin{cases} (-1)^\ell\binom{m-1}{\ell-1} & \text{if } \ell \le n-1 \\ (-1)^\ell\binom{m-1}{\ell-1}-(-1)^n\binom{m+n-1}{m} & \text{if } \ell = n \end{cases}$$

using problems 3 and 9 of chapter III of Feller volume I [6]. Thus (3.31) becomes

$$(3.33)\ \zeta_{(n,a,m)} = \frac{n}{1-p_{(n,a)}}\binom{m+n-1}{m}^{-1}\sum_{\ell=1}^{n}\frac{1}{\ell}\binom{n-1}{\ell-1}\binom{m-1}{\ell-1}(1-\ell a)_+^{m+n-1} - \frac{(1-na)_+^{m+n-1}}{1-p_{(n,a)}}\ .$$

This is seen to be equal to (3.27), completing the proof. ||

Lemma 3: The density of $B_{(n,a)}$ is $f_{(n,a)}(t)$. In particular, $B_{(n,a)}$ is a continuous random variable.

Proof: From lemma 2, we see that

$$(3.34)\qquad E\,B^m_{(n,a)} = \int_0^1 t^m f_{(n,a)}(t)dt \qquad m = 0,1,2,\ldots$$

Because a probability distribution on $[0,1]$ is uniquely characterized by its moments (theorem 1 of section VII.3 of Feller volume II [6],) we would be done if we knew $f_{(n,a)}$ to be nonnegative. This is difficult to ascertain directly from (3.21), although $f_{(n,a)}$ is certainly bounded below. Fix n and a, and choose any $\theta > 0$

satisfying $\theta + f_{(n,a)}(t) \geq 0$ for all $t \in [0,1]$. Define a true probability density function

$$(3.35) \qquad g_{(n,a)}(t) = \frac{\theta + f_{(n,a)}(t)}{\theta + 1} .$$

Let $P_{(n,a)}$ denote the probability measure induced by $B_{(n,a)}$ on $[0,1]$, and let μ denote Lebesgue measure. Define the probability measure $\nu_{(n,a)}$ on $[0,1]$ by

$$(3.36) \quad \nu_{(n,a)}(K) = \frac{\theta\mu(K) + P_{(n,a)}(K)}{\theta + 1} , \quad \text{all measurable} \quad K \subset [0,1] .$$

The moments of $g_{(n,a)}$ and of $\nu_{(n,a)}$ are easily seen to be equal using lemma 2. Thus $g_{(n,a)}$ is the density of $\nu_{(n,a)}$ and it follows from

$$(3.37) \quad \frac{\theta\mu(K) + P_{(n,a)}(K)}{\theta + 1} = \nu_{(n,a)}(K) = \int_K g_{(n,a)}(t)dt = \frac{\theta\mu(K) + \int_K f_{(n,a)}(t)dt}{\theta + 1}$$

for all measurable K, that $f_{(n,a)}$ is the density of $P_{(n,a)}$ and hence of $B_{(n,a)}$. ||

Lemma 4: The cumulative distribution function of $D_{(n,a)}$ is $F_{(n,a)}(t)$ given by (3.24).

Proof: Let $F_{(n,a)}(t) = P(D_{(n,a)} \leq t)$ be the cumulative distribution function of $D_{(n,a)}$. From lemmas 1 and 3, it follows that

$$(3.38) \quad F_{(n,a)}(t) = p_{(n,a)} I_{\{t \geq (1-na)_+\}} + (1 - p_{(n,a)}) \int_0^t f_{(n,a)}(t)dt$$

where I_X denotes the indicator function of the set X. For convenience, we will drop the subscripts (n,a) for the rest of the proof; because n and a are fixed, this will cause no problems. It is convenient to rewrite (3.21) as

$$(3.39) \qquad f(t) = \frac{n}{1-p} \sum_{\ell=1}^{n} \frac{1}{\ell!} (-1)^{\ell} \binom{n-1}{\ell-1} \frac{d^{\ell}}{dt^{\ell}} [t^{\ell-1}(1-\ell a-t)_+^{n-1}] \; .$$

That this is equivalent to (3.21) is seen by expanding the derivative of the product by Leibniz's rule. Using the form (3.39), the integral in (3.38) is easily done. We need only replace $\frac{d^{\ell}}{dt^{\ell}}$ by $\frac{d^{\ell-1}}{dt^{\ell-1}}$ and expand again by Leibniz's rule to obtain a piecewise primitive (indefinite integral) F_0 of f. There may, however, be discontinuities at $t = (1-\ell a)_+$, $\ell = 1,\ldots,n$. We find that

$$(3.40) \qquad F_0(t) = \frac{1}{1-p} \sum_{\ell=1}^{n} (-1)^{\ell} \binom{n}{\ell} \sum_{k=0}^{\ell-1} (-1)^k \binom{\ell-1}{k} \binom{n-1}{k} t^k (1-\ell a-t)_+^{n-k-1}$$

which is continuous except at $t = (1-na)_+$, because when $\ell = n$ and $k = n-1$, we have $(1-\ell a-t)_+^{n-k-1} = I_{\{t < (1-na)_+\}}$. Adding a constant to F_0 when $t < (1-na)_+$ will yield a true continuous primitive of f, namely

$$(3.41) \qquad F_1(t) = F_0(t) + \frac{1}{1-p} (1-na)_+^{n-1} \, I_{\{t < (1-na)_+\}} \; .$$

We may use this to calculate

$$\int_0^t f(t)dt = F_1(t)-F_1(0) = F_0(t)-F_0(0) - \frac{1}{1-p}(1-na)_+^{n-1} I_{\{t \geq (1-na)_+\}}$$

$$(3.42) \qquad = \frac{1}{1-p} \sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell} \sum_{k=0}^{\ell-1} (-1)^k \binom{\ell-1}{k}\binom{n-1}{k} t^k (1-\ell a-t)_+^{n-k-1}$$

$$- \frac{1}{1-p} \sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell}(1-\ell a)_+^{n-1} - \frac{1}{1-p}(1-na)_+^{n-1} I_{\{t \geq (1-na)_+\}} \; .$$

Note that we may write

$$(3.43) \qquad - \sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell}(1-\ell a)_+^{n-1} = 1-p \; I_{\{a > 1/n\}} \; .$$

Using (3.42) and (3.43) in (3.38) we get

$$(3.44) \quad F(t) = 1 + \sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell} \sum_{k=0}^{\ell=1} (-1)^k \binom{\ell-1}{k}\binom{n-1}{k} t^k (1-\ell a-t)_+^{n-k-1}$$

$$- p \; I_{\{a > 1/n\}} + [p-(1-na)_+^{n-1}] \; I_{\{t \geq (1-na)_+\}} \; .$$

It is easy to see that

$$(3.45) \qquad p-(1-na)_+^{n-1} = p \; I_{\{a > 1/n\}} \; .$$

Since $a > 1/n$ implies $(1-na)_+ = 0$, for $t \geq 0$ we have

$$(3.46) \qquad [p-(1-na)_+^{n-1}] \; I_{\{t \geq (1-na)_+\}} = p \; I_{\{a > 1/n\}}$$

so that (3.44) reduces to (3.24), completing the proof. ||

Corollary: The cumulative distribution function of the coverage $C_{(n,a)}$ of n random arcs of length a on a circle is

$$(3.47) \quad G_{(n,a)}(t) = P(C_{(n,a)} \leq t) \quad \sum_{\ell=1}^{n} \sum_{k=0}^{n-1} (-1)^{k+\ell+1} \binom{n}{\ell} \binom{\ell-1}{k} \binom{n-1}{k} (1-t)^k (t-\ell a)_+^{n-k-1}$$

Proof: This is immediate from theorem 3 and the relation $C_{(n,a)} = 1 - D_{(n,a)}$. ||

Theorem 4: The limiting distribution of $nB_{(n,a)}$ for fixed a, as n tends to infinity is the exponential distribution. We have

$$(3.48) \qquad \lim_{n \to \infty} P\left(\frac{nB_{(n,a)}}{1-a} \leq t\right) = 1 - e^{-t}$$

and we also have

$$(3.49) \qquad P\left(\frac{nD_{(n,a)}}{1-a} > t\right) \sim n(1-a)^{n-1} e^{-t}$$

for each fixed t, as n tends to infinity.

Proof: (3.48) is established using the method of moments. We assume $n > 1/a$ so that, from (3.23)

$$(3.50) \qquad p_{(n,a)} = \sum_{\ell=0}^{b} (-1)^{\ell} \binom{n}{\ell} (1-\ell a)^{n-1}$$

where $b = [1/a]$, the greatest integer contained in $1/a$. Note also that $(1-na)_+ = 0$ in this case, so that using the decomposition of theorem 3 we have

$$(3.51)\qquad E\left(\frac{nB_{(n,a)}}{1-a}\right)^m = \frac{1}{1-p_{(n,a)}}\,\frac{n^m}{(1-a)^m}\,E\,D^m_{(n,a)}$$

Substituting for $E\,D^m_{(n,a)}$ from theorem 2 and factoring powers of $(1-a)$ we may write this as

$$(3.52)\qquad \begin{aligned} E\left(\frac{nB_{(n,a)}}{1-a}\right)^m &= \left[n^m \binom{m+n-1}{m}^{-1}\right] \cdot \\ &\left[1 - \sum_{\ell=2}^{b} \frac{(-1)^\ell}{n}\binom{n}{\ell}\left(\frac{(1-\ell a)_+}{1-a}\right)^{n-1}\right]^{-1} \\ &\cdot \left[1 + \sum_{\ell=2}^{b} \frac{1}{\ell}\binom{m-1}{\ell-1}\binom{n-1}{\ell-1}\left(\frac{(1-\ell a)_+}{1-a}\right)^{m+n-1}\right] . \end{aligned}$$

Holding m and a fixed, we take limits in (3.52) as n tends to infinity. The first bracketed term satisfies

$$(3.53)\qquad \lim_{n\to\infty} n^m \binom{m+n-1}{m}^{-1} = \lim_{n\to\infty} m! \prod_{i=0}^{m-1} \frac{n}{n+i} = m! \quad .$$

For each $\ell \geq 2$, we have $0 \leq (1-\ell a)_+/(1-a) < 1$ so that

$$(3.54)\qquad \lim_{n\to\infty} \binom{n}{\ell}\left(\frac{(1-\ell a)_+}{1-a}\right)^{n-1} = 0 \quad .$$

Because of this and the fact that the ranges of summation in (3.52) do not depend on n, we have

$$(3.55)\qquad \lim_{n\to\infty} E\left(\frac{nB_{(n,a)}}{1-a}\right)^m = m!$$

These are the moments of the exponential distribution. To show convergence in distribution, we apply the condition in section 8.12 of L. Breiman [2]: We must show that

$$\overline{\lim_{m \to \infty}} \frac{(m!)^{1/m}}{m} < \infty \tag{3.56}$$

so that the moments do not grow too quickly. (3.56) is easily seen to be the case, because

$$\frac{(m!)^{1/m}}{m} = (m!/m^m)^{1/m} < 1 \quad . \tag{3.57}$$

To establish (3.49), observe that

$$P(\frac{nD_{(n,a)}}{1-a} > t) = (1-p_{(n,a)})P(\frac{nB_{(n,a)}}{1-a} > t) \tag{3.58}$$

and

$$\lim_{n \to \infty} \frac{1-p_{(n,a)}}{n(1-a)^{n-1}} = \lim_{n \to \infty} (1 - \sum_{\ell=2}^{[1/a]} \frac{(-1)^{\ell}}{n} \binom{n}{\ell} (\frac{(1-\ell a)_+}{1-a})^{n-1}) = 1 \ . \tag{3.59}$$

Using (3.58), (3.59), and (3.48) we have

$$\lim_{n \to \infty} P(\frac{nD_{(n,a)}}{1-a} > t)/(n(1-a)^{n-1}e^{-t}) = 1 \tag{3.60}$$

completing the proof. $\|$

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Place n arcs of equal lengths randomly on the circumference of a circle, and let C denote the proportion covered. The moments of C (moments of coverage) are found by solving a recursive integral equation, and a formula is derived for the cumulative distribution function. The asymptotic distribution of C for large n is explored, and is shown to be related to the exponential distribution.

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